Errata to Real Analysis by N. L. Carothers, 2000

These issues were reported by Professor Rupert Frank while teaching Ma108a at Caltech during Fall 2013. I record them here and provide counterexamples. I also emailed them to the author.

- Chapter 2, page 29, exercise 26: $f : \Delta \to [0,1]$ is the Cantor function and $x, y \in \Delta$ with $x < y$. "If $f(x) = f(y)$, show that x has two distinct binary decimal representations" should instead read "show that x has two distinct *ternary* decimal representations." As a counterexample to the stated exercise, consider $x = \frac{1}{3}, y = \frac{2}{3}$. Then, $x, y \in \Delta$ with $x < y$, and $f(x) = f(y) = \frac{1}{2}$. Yet, $x = \frac{1}{3}$ has only one binary decimal representation.

- Chapter 3, page 39, exercise 10, part (ii). The first part of the exercise shows that $d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on H^∞. In the second part, we take $x, y \in H^\infty$ and $k \in \mathbb{N}$, and let $M_k = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$. We are directed to "show that $2^{-k}M_k \le d(x,y) \le M_k + 2^{-k}$." The upper bound is incorrect; we suggest that it instead read "$2^{-k}M_k \le d(x,y) \le M_k + 2^{-k+1}$." As a counterexample to the stated exercise, take $x = (x_n)$ defined by $x_1 = 0$ and $x_n = 1$ for $n > 1$ and $y = (y_n)$ defined by $y_1 = 0$ and $y_n = -1$ for $n > 1$, and take $k = 1$. Then, $M_k = \max\{|x_1 - y_1|\} = 0$ and $2^{-k} = \frac{1}{2}$, so, according to the stated exercise, we would have $d(x,y) \le \frac{1}{2}$. Yet, $d(x,y) = 0 + \sum_{n=2}^{\infty} 2^{-n}|1 - -1| = 1 \not\le \frac{1}{2}$.

  Let us show that our suggested upper bound of 2^{-k+1} is satisfactory: $d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| = \sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n=k+1}^{\infty} 2^{-n}|x_n - y_n| \le \sum_{n=1}^{k} 2^{-n}M_k + \sum_{n=k+1}^{\infty} 2^{-n+1} = (1 - 2^{-k})M_k + 2^{-k+1} \le M_k + 2^{-k+1}$.

- Chapter 11, page 181, theorem 11.18: The proof of one direction of the Arzel'a-Ascoli theorem is flawed. We assume that X is a compact metric space and $\mathcal{F}$ is a closed, uniformly bounded, and equicontinuous subset of $C(X)$, the space of all continuous real-valued functions on X. We wish to show that $\mathcal{F}$ is compact. The text's approach is to let (f_n^o) be any sequence in $\mathcal{F}$, and show that (f_n^o) contains a subsequence (f_n) that is uniformly Cauchy. (The text does not use (f_n^o) in its notation; instead, it begins by letting (f_n) refer to an arbitrary sequence from $\mathcal{F}$, then re-uses (f_n) to refer to a subsequence of the original sequence.) To use this approach, it is necessary to show that for any choice of (f_n^o), there is a subsequence (f_n) such that for all $\epsilon > 0$, there is an N such that for any $x \in X$ and any $m, n \ge N$, we have $|f_m(x) - f_n(x)| < \epsilon$. Importantly, the subsequence (f_n) must not depend on the value of ϵ. In the text's proof, however, the choice of subsequence depends on the choice of finite δ-net, and the choice of δ depends on ϵ, so the text's choice of subsequence depends on ϵ. So, the text does not really show that (f_n) is uniformly Cauchy.

  The following is Professor Frank's approach to showing that any sequence (f_n) from $\mathcal{F}$ has a uniformly Cauchy subsequence. Since X is compact, it is separable. Let (x_j) be a dense, countable subset of X. Since $(f_k(x_1))$ is a bounded sequence of reals, a subsequence converges; call it $(f_{k_n^{(1)}}(x_1))$. Since $(f_{k_n^{(1)}}(x_2))$ is a bounded sequence of reals, a subsequence converges; call it $(f_{k_n^{(2)}}(x_2))$. Continue in this manner. Now, consider the diagonal sequence $(f_{k_n^{(n)}})$. Observe that $(f_{k_n^{(n)}}(x_j))$ converges for any fixed j. We claim that $(f_{k_n^{(n)}})$ is the desired Cauchy sequence in $C(X)$. Fix $\epsilon > 0$. By the equicontinuity of $\mathcal{F}$, we may choose a $\delta > 0$ such that whenever $x, y \in X$ satisfy $d(x,y) < \delta$, we have $|f(x) - f(y)| < \epsilon/3$. By compactness of X, $X = \bigcup_{i=1}^{m} B_{\delta/2}(y_i)$ for some $y_1, \ldots, y_m \in X$. Since (x_j) is dense, there are $x_{j_1}, \ldots, x_{j_m}$ such that $d(x_{j_i}, y_i) < \delta/2$ for $i = 1, \ldots, m$. Now let $x \in X$ and choose $i \in \{1, \ldots, m\}$ such that $x \in B_{\delta/2}(y_i)$. Note that $|x - x_{j_i}| < \delta$, so $|f_k(x) - f_k(x_{j_i})| < \epsilon/3$ for any k. And, by construction, there exists an N not depending on x such that for all $n, n' \ge N$ and for all $i = 1, \ldots, m$, we have that $|f_{k_n^{(n)}}(x_{j_i}) - f_{k_{n'}^{(n')}}(x_{j_i})| \leq \epsilon/3$. Thus, for $n, n' \geq N$, we have $|f_{k_n^{(n)}}(x) - f_{k_{n'}^{(n')}}(x)| \leq |f_{k_n^{(n)}}(x) - f_{k_n^{(n)}}(x_{j_i})| + |f_{k_n^{(n)}}(x_{j_i}) - f_{k_{n'}^{(n')}}(x_{j_i})| + |f_{k_{n'}^{(n')}}(x_{j_i}) - f_{k_{n'}^{(n')}}(x)| = \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

- Chapter 11, page 182, exercise 57: In this exercise, we take (f_n) to be a sequence of differentiable functions from $[a, b]$ to $\mathbb{R}$ satisfying $|f_n'(x)| \leq 1$ for all n and x. We are directed to show that some subsequence of (f_n) is uniformly convergent, but this is not necessarily true. For example, take (f_n) defined by $f_n(x) = \frac{x}{2} + n$. Then, each f_n is differentiable with $|f_n'(x)| = \frac{1}{2} < 1$. Yet, $||f_n - f_m||_\infty \geq 1$ for all $n \neq m$. Hence, there is no uniformly convergent subsequence of (f_n).